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On Functions Analogous to the Theta-Functions.

Dissertation

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by

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Introduction.

M. Appell in a brief note in the "Annales de la Faculté des Sciences de Marseilles", gives as an example of a function of three variables having a true period and a quasi-period, analogous to the Θ-functions, the function

$$\varphi(x, y, z) = \sum_{m=-\infty}^{m=\infty} e^{am^4 + 4xm^3 + 6ym^2 + 4zm}$$

when the real part of a is to be negative.

This function evidently satisfies the conditions

$$\varphi\left(x+\tfrac{\pi i}{2}, y, z\right) = \varphi\left(x, y+\tfrac{\pi i}{3}, z\right) = \varphi\left(x, y, z+\tfrac{\pi i}{2}\right) = \varphi(x, y, z)$$

$$\varphi(x+a, y+2x+a, z+3x+3y+a)$$
$$= e^{-(a+4x+6y+4z)}\,\varphi(x, y, z)$$

$$6\frac{\partial\varphi}{\partial x} = \frac{\partial^2\varphi}{\partial y\,\partial z} \qquad 8\frac{\partial\varphi}{\partial y} = 3\frac{\partial^2\varphi}{\partial z^2}$$

Moreover, as M. Appell shows, these conditions are sufficient to determine $\varphi(x, y, z)$ to within a constant factor.

The object of the present paper is to investigate the properties of this function

and of functions derived from it, as well as of others similar to it, pointing out as far as possible their analogy to those of the Θ-functions. As is not surprising, some of the properties of the latter seem to have no analogues in the case of the functions here considered. In such instances, it has been endeavored to assign the reason as far as possible.

[illegible] [illegible] [illegible] [illegible] Craig [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] consequence [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] nature.

[illegible] this, the assistance rendered me by Professor Craig, at whose suggestion this subject was selected, was invaluable; my appreciation of which [illegible] [illegible] [illegible]

those of ordinary theses. I desire also to acknowledge the debt of gratitude I owe to both Professor Craig and Professor Franklin for their interest manifested in many ways throughout my entire connection with the Johns Hopkins University.

Let $f(x, y, z)$ be a holomorphic function satisfying the conditions

$$(1) \quad f(x+\omega_1, y, z) = f(x, y+\omega_2, z) = f(x, y, z+\omega_3)$$
$$= f(x, y, z)$$

$$(2) \quad f\left(x + \frac{2\omega_1 a}{\pi i} p_1,\ y + \frac{3\omega_2 x}{\omega_1} p_1 + \frac{3\omega_2 a}{\pi i} p_1^2,\ z + \frac{2\omega_3 y}{\omega_2} p_1 + \frac{3\omega_3 x}{\omega_1} p_1^2 + \frac{\omega_3 a}{\pi i} p_1^3\right)$$
$$= e^{-a p_1^4 - 2\pi i\left(\frac{x}{\omega_1} p_1^3 + \frac{y}{\omega_2} p_1^2 + \frac{z}{\omega_3} p_1\right) b} f(x, y, z)$$

$$(3) \quad \begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\omega_2 \omega_3}{2\pi i \omega_1} \dfrac{\partial^2 f}{\partial y^2} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{\omega_3^2}{2\pi i} \dfrac{\partial^2 f}{\partial z^2} \end{cases}$$

where $\omega_1, \omega_2, \omega_3$ are any quantities, real or imaginary.

b any given integer

a a constant whose real part is negative.

The most general entire function of x, y, z satisfying conditions (1) is given by the Fourier series

$$(4) \quad f(x, y, z) = \sum_{n=-\infty}^{n=\infty} \sum_{l=-\infty}^{l=\infty} \sum_{m=-\infty}^{m=\infty} C_{n,l,m}\, e^{2\pi i\left(\frac{x}{\omega_1} n + \frac{y}{\omega_2} l + \frac{z}{\omega_3} m\right)}$$

where the coefficient $C_{k,l,m}$ is independent of x, y, z. In order that $f(x, y, z)$ also satisfy condition (3) we must have

$$k = lm \quad \text{and} \quad l = m^2$$

$$\text{or} \quad k = m^3 \qquad l = m^2$$

and we now have only the simple infinite series

$$(4) \quad f(x, y, z) = \sum_{m=-\infty}^{m=\infty} C_m e^{2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)}$$

Finally, from (2) we have, on multiplying both sides of the equation by

$$e^{am^4 + ap^4 + 2\pi i\left(\frac{x}{\omega_1} p^3 + \frac{y}{\omega_2} p^2 + \frac{z}{\omega_3} p\right)}$$

and properly collecting the terms,

$$\sum_m C_m e^{a(m+p)^4 + 2\pi i\left[\frac{x}{\omega_1}(m+p)^3 + \frac{y}{\omega_2}(m+p)^2 + \frac{z}{\omega_3}(m+p)\right]}$$

$$= \sum_m e^{am^4 + 2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)}$$

In order that this equation be satisfied, it is evidently necessary and sufficient that for all values of m

$$C_m = C_{m+p}.$$

Hence the most general function of x, y, z satisfying the conditions (1), (2), (3)

will be given by

$$(5)\qquad f(x,y,z) = \sum_m C_m e^{am^4+2\pi i\left(\frac{x}{\omega_1}m^3+\frac{y}{\omega_2}m^2+\frac{z}{\omega_3}m\right)}$$

with
$$C_m = C_{m+p}$$

Since by hypothesis, the real part of a is negative, this function is holomorphic for all values of $x\ y\ z$.

If we write

$$(6)\quad \begin{cases} R_1(x,y,z) = \displaystyle\sum_{\kappa=-\infty}^{\kappa=\infty} e^{a(\kappa p+1)^4+2\pi i\left[\frac{x}{\omega_1}(\kappa p+1)^3+\frac{y}{\omega_2}(\kappa p+1)^2+\frac{z}{\omega_3}(\kappa p+1)\right]} \\ R_2(x,y,z) = \displaystyle\sum e^{a(\kappa p+2)^4+2\pi i\left[\frac{x}{\omega_1}(\kappa p+2)^3+\frac{y}{\omega_2}(\kappa p+2)^2+\frac{z}{\omega_3}(\kappa p+2)\right]} \\ \text{-------------------------} \\ R_\rho(x,y,z) = \displaystyle\sum e^{a(\kappa p+\rho)^4+2\pi i\left[\frac{x}{\omega_1}(\kappa p+\rho)^3+\frac{y}{\omega_2}(\kappa p+\rho)^2+\frac{z}{\omega_3}(\kappa p+\rho)\right]} \\ \text{-------------------------} \\ R_p(x,y,z) = \displaystyle\sum e^{a(\kappa p)^4+2\pi i\left[\frac{x}{\omega_1}(\kappa p)^3+\frac{y}{\omega_2}(\kappa p)^2+\frac{z}{\omega_3}(\kappa p)\right]} \end{cases}$$

it is clear that $f(x,y,z)$ will be a linear homogeneous function of

$$R_1,\ R_2,\ \cdots\cdots,\ R_\rho,\ \cdots\cdots,\ R_p$$

Moreover, the latter are linearly independent as may be seen at once from their [illegible]! (They can however be

replaced by simpler functions.

Write

$$(7)\qquad \varphi(x,y,z) = \sum_m e^{am^4 + 2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)}$$

Then for λ, μ, ν any integers, we have

$$\varphi\left(x + \frac{\lambda\omega_1}{p}, y + \frac{\mu\omega_2}{p}, z + \frac{\nu\omega_3}{p}\right)$$

$$= \sum_m e^{am^4 + 2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right) + \frac{2\pi i}{p}\left(\lambda m^3 + \mu m^2 + \nu m\right)}$$

or

$$(8)\qquad \varphi\left(x + \frac{\lambda\omega_1}{p}, y + \frac{\mu\omega_2}{p}, z + \frac{\nu\omega_3}{p}\right) = e^{\frac{2\pi i}{p}(\lambda + \mu + \nu)} R_1 + e^{\frac{2\pi i}{p}(8\lambda + 4\mu + 2\nu)} R_2 + \cdots + e^{\frac{2\pi i}{p}(\rho^3\lambda + \rho^2\mu + \rho\nu)} R_\rho + \cdots + R_p$$

Giving to λ, μ, ν, each separately, all integer values from 0 to $p-1$, we get p^3 equations of the type (8) to be satisfied by the p quantities R. Of these p^3 equations only p can be independent. Moreover there are p independent ones among them, viz., as we shall see, those obtained by putting $\lambda = \mu = 0$ and letting ν [illegible]

[illegible] = 0, ... p − 1, [illegible]

$$(3)\quad \varphi\left(x, y, z + \frac{\nu\omega_3}{p}\right) = e^{[illegible]} R_1 + \cdots\cdots + e^{[illegible]} R_\rho + \cdots\cdots + R_p$$

$$\nu = 0, 1, \ldots\ldots p - 1.$$

For the sake of brevity we shall introduce the following notation.—

Write

$$\varphi(x, y, z) = [0, 0, 0]$$

$$\varphi\left(x + \frac{\lambda\omega_1}{p}, y, z\right) = [\lambda, 0, 0]$$

$$\varphi\left(x, y + \frac{\mu\omega_2}{p}, z\right) = [0, \mu, 0]$$

$$\varphi\left(x, y, z + \frac{\nu\omega_3}{p}\right) = [0, 0, \nu]$$

$$\varphi\left(x + \frac{\lambda\omega_1}{p}, y + \frac{\mu\omega_2}{p}, z\right) = [\lambda, \mu, 0]$$

- -

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$$\varphi\left(x + \frac{\lambda\omega_1}{p}, y + \frac{\mu\omega_2}{p}, z + \frac{\nu\omega_3}{p}\right) = [\lambda, \mu, \nu]$$

Also write

$$[0, 0, 1] = \zeta_1(x, y, z) = \zeta_1$$

Putting

$$\gamma_1 = e^{\frac{2\pi i}{p}}, \quad \gamma_2 = e^{[illegible]} \cdots\cdots\cdots, \quad \gamma_\rho = e^{[illegible]} \cdots\cdots \gamma_p = 1$$

our equations (3) may then be written [illegible]

$$
(10)\quad \begin{cases}
[0,0,0] \equiv \xi_0 = R_1 + R_2 + \cdots\cdots + R_\rho + \cdots\cdots + R_p \\
[0,0,1] \equiv \xi_1 = \gamma_1 R_1 + \gamma_2 R_2 + \cdots\cdots + \gamma_\rho R_\rho + \cdots\cdots + \gamma_p R_p \\
[0,0,2] \equiv \xi_2 = \gamma_1^2 R_1 + \gamma_2^2 R_2 + \cdots\cdots + \gamma_\rho^2 R_\rho + \cdots\cdots + \gamma_p^2 R_p \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
[0,0,p-1] \equiv \xi_{p-1} = \gamma_1^{p-1} R_1 + \gamma_2^{p-1} R_2 + \cdots\cdots + \gamma_\rho^{p-1} R_\rho + \cdots\cdots + \gamma_p^{p-1} R_p
\end{cases}
$$

The independence of these equations follows at once from the fact that the determinant of the system

$$
\Delta = \begin{vmatrix}
1 & 1 & \cdots\cdots & 1 & \cdots\cdots & 1 \\
\gamma_1 & \gamma_2 & \cdots\cdots & \gamma_\rho & \cdots\cdots & \gamma_p \\
\gamma_1^2 & \gamma_2^2 & \cdots\cdots & \gamma_\rho^2 & \cdots\cdots & \gamma_p^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_1^{p-1} & \gamma_2^{p-1} & \cdots\cdots & \gamma_\rho^{p-1} & \cdots\cdots & \gamma_p^{p-1}
\end{vmatrix}
$$

$$
= \prod^{i\,j} (\gamma_i - \gamma_j) \qquad i, j = 1, 2, \cdots, p \qquad i \neq j
$$

[illegible]

γ's being the p^{th} roots of unity. Of course the value of Δ^2 is $(-1)^{\frac{(p-1)(p-2)}{2}} p^p$.

Denoting the minor of [illegible] in Δ by $C_{\lambda,\mu}$ [illegible] the determinant of the minors is

$$D = \begin{vmatrix} C_{0,0} & C_{1,0} & \cdots\cdots & C_{p-1,0} \\ C_{0,1} & C_{1,1} & \cdots\cdots & C_{p-1,1} \\ C_{0,2} & C_{1,2} & \cdots\cdots & C_{p-1,2} \\ \cdots & \cdots & \cdots & \cdots \\ C_{0,p-1} & C_{1,p-1} & \cdots\cdots & C_{p-1,p-1} \end{vmatrix}$$

Solving [illegible] equations (10) we get

$$(11)\quad \begin{cases} \Delta R_1 = C_{0,0}\xi_0 + C_{0,1}\xi_1 + C_{0,2}\xi_2 + \cdots\cdots + C_{0,p-1}\xi_{p-1} \\ \Delta R_2 = C_{1,0}\xi_0 + C_{1,1}\xi_1 + C_{1,2}\xi_2 + \cdots\cdots + C_{1,p-1}\xi_{p-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta R_p = C_{p-1,0}\xi_0 + C_{p-1,1}\xi_1 + C_{p-1,2}\xi_2 + \cdots\cdots + C_{p-1,p-1}\xi_{p-1} \end{cases}$$

The remaining $p^3 - p$ functions

$$[\lambda, \mu, \nu]$$

$$\lambda, \mu, \nu = 0, 1, \cdots\cdots, p-1 \quad \text{but } \lambda, \mu \neq 0, 0$$

can now be expressed as linear homogeneous functions —

λ $\quad\quad = 0, 1, \ldots\ldots, p-1$

From (6) it is [illegible] that

$$
(12)\quad \begin{cases}
R_1\left(x + \frac{\lambda\omega_1}{p}, y, z\right) = e^{\lambda\frac{2\pi i}{p}} R_1(x, y, z) = \gamma_\lambda R_1 \\
\text{------------------------------} \\
R_\sigma\left(x + \frac{\lambda\omega_1}{p}, y, z\right) = e^{\lambda\sigma\frac{2\pi i}{p}} R_\sigma(x, y, z) = \gamma_\lambda^\sigma R_\sigma \\
\text{------------------------------} \\
R_p\left(x + \frac{\lambda\omega_1}{p}, y, z\right) = e^{\lambda p\frac{2\pi i}{p}} R_p(x, y, z) = R_p
\end{cases}
$$

Making these changes in (11) we get the following p systems of $p-1$ equations each:-

$$(13_1)\quad \gamma_\lambda \Delta R_1 = C_{0,0}[\lambda,0,0] + C_{0,1}[\lambda,0,1] + \cdots\cdots + C_{0,p-1}[\lambda,0,p-1]$$

$$(13_\sigma)\quad \gamma_\lambda^\sigma \Delta R_\sigma = C_{\sigma-1,0}[\lambda,0,0] + C_{\sigma-1,1}[\lambda,0,1] + \cdots\cdots + C_{\sigma-1,p-1}[\lambda,0,p-1]$$

$$(13_p)\quad \Delta R_p = C_{p-1,0}[\lambda,0,0] + C_{p-1,1}[\lambda,0,1] + \cdots\cdots + C_{p-1,p-1}[\lambda,0,p-1]$$

$$\lambda = 1, 2, \ldots\ldots, p-1$$

[illegible] the systems of p equations obtained by taking the λth equation of each of the [illegible]

[illegible] D [illegible]

[illegible] Δ [illegible]

[illegible] the [illegible] quantities

$$[\lambda, o, \nu] \qquad \begin{matrix} \lambda = [illegible], p-1 \\ \nu = 0, 1, 2, \cdots\cdots, p-1 \end{matrix}$$

in terms of $R_1, R_2, \cdots\cdots, R_\rho, \cdots, R_p$ which in turn can be expressed linearly in terms of

[illegible]

$$[illegible] \qquad = 0, 1, 2, \cdots\cdots, p-1$$

Thus taking the system

$$\begin{cases} \gamma_\lambda [illegible] \Delta R_1 = C_{0,0}[\lambda, 0, 0] + C_{0,1}[\lambda, 0, 1] + \cdots\cdots + C_{0,p-1}[\lambda, 0, p-1] \\ \gamma_\lambda^{8} \Delta R_2 = C_{1,0}[\lambda, 0, 0] + C_{1,1}[\lambda, 0, 1] + \cdots\cdots + C_{1,p-1}[\lambda, 0, p-1] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma_\lambda^{[illegible]} \Delta R_{[illegible]} = C_{[illegible]}[\lambda, 0, 0] + C_{[illegible]}[\lambda, 0, 1] + \cdots\cdots + C_{[illegible], p-1}[\lambda, 0, p-1] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta R_p = C_{p-1,0}[\lambda, 0, 0] + C_{p-1,1}[\lambda, 0, 1] + \cdots\cdots + C_{p-1,p-1}[\lambda, 0, p-1] \end{cases}$$

and remembering that the minor of $C_{j,i}$ in D is $\gamma_{j+}^{\lambda}\, \Delta^{p-2}$ we have

$$[\lambda, 0, \nu] = \sum_{j=1}^{j=[illegible]} [illegible] R_j$$

[illegible] (91)

$$(14) \qquad \Delta[\lambda,0,\nu] = \sum_{\rho=1}^{\rho=p} \sum_{j=0}^{j=p-1} \gamma_\lambda^{\rho} \gamma_\rho^{\nu} C_{\rho-1,j}\, \xi_j$$

$$\lambda = 1, 2, \ldots\ldots, p-1$$

$$\nu = 0, 1, 2, \ldots\ldots, p-1$$

In exactly the same way we get

$$(15) \qquad \Delta[0,\mu,\nu] = \sum_{\rho=1}^{\rho=p} \sum_{j=0}^{j=p-1} \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,j}\, \xi_j$$

$$\mu = 1, 2, \ldots\ldots, p-1$$

$$\nu = 0, 1, 2, \ldots\ldots, p-1$$

or

$$\Delta[0,\mu,\nu] = \sum_{\rho=1}^{\rho=p} \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,0}[0,0,0] + \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,1}[0,0,1] + \ldots$$
$$+ \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,m}[0,0,m] + \cdots + \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,p-1}[0,0,p-1]$$

$$\therefore \Delta[\lambda,\mu,\nu] = \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,0}[\lambda,0,0] + \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,1}[\lambda,0,1]$$
$$+ \cdots\cdots\cdots + \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,p-1}[\lambda,0,p-1]$$

whence from (14) we have

$$(16) \quad \Delta^2[\lambda,\mu,\nu] = \sum_{\rho=1}^{\rho=p} \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,0} \sum_{\rho'=1}^{\rho'=p} \sum_{j=0}^{j=p-1} \gamma_\lambda^{\rho'} C_{\rho'-1,j}\, \xi_j$$
$$+ \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,1} \sum_{\rho'} \sum_j \gamma_\lambda^{\rho'} \gamma_{\rho'}^{1} C_{\rho'-1,j}\, \xi_j$$
$$+ \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,2} \sum_{\rho'} \sum_j \gamma_\lambda^{\rho'} \gamma_{\rho'}^{2} C_{\rho'-1,j}\, \xi_j$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \sum_\rho \gamma_\mu^{\rho^2} \gamma_\rho^{\nu} C_{\rho-1,p-1} \sum_{\rho'} \sum_j \gamma_\lambda^{\rho'} \gamma_{\rho'}^{p-1} C_{\rho'-1,j}\, \xi_j$$

where $\mu = 1, 2, \ldots, p$ [illegible]

$\nu = 1, 2, \ldots, p-1$

If we write [illegible] in the form

[illegible] $[\lambda, 0, \nu] = \frac{1}{\Delta} \sum_{\rho=1} \sum_{j} \gamma$ [illegible] $c_{\ldots, j}\, \xi_j$

$\equiv \frac{1}{\Delta} \sum_{j=} A^{\lambda}$ [illegible] ξ_j

The determinant of [illegible] of the [illegible] p equations, obtained by keeping λ fixed and allowing ν to take successively all integer values from 0 to $p-1$, is seen at once [illegible]

$$[illegible]_{\lambda}^{(1^3+2^3+\cdots+p^3)} \frac{D\,\Delta}{\Delta^p}$$

[illegible] $D = \Delta^{p-}$ [illegible]

$$[illegible] \equiv \frac{[illegible]}{\Delta^p} \left| A^{\mu} \right|$$

Similarly, writing (10) in the form

(8) $[0, \mu, \nu] = \frac{1}{\Delta} \sum_{\rho} \sum_{j} \gamma_{\mu}^{\rho} \gamma_{\rho}^{\nu}\, c_{\rho-1, j}\, \xi_j$

$\equiv \frac{1}{\Delta} \sum_{j} B_{j}^{\mu}\, \xi_j$

the determinant of any of the systems of p equations obtained by keeping μ fixed is found to be

$$\gamma_\mu^{(1^2+2^2+\cdots\cdots+p^2)} = \gamma_\mu^{\frac{p(p-1)(2p-1)}{6}} \equiv \frac{1}{\Delta^p}\left|B^{(\mu)}_{j,\kappa}\right|$$

Finally (6) may be put in the form

$$(9) \qquad \left[\lambda,\mu,\nu\right] = \frac{1}{\Delta^2} \sum_{\kappa=0}^{\kappa=p-1} \sum_{j=0}^{j=p-1} A^{(\lambda)}_{\kappa,\nu}\, B^{(\mu)}_{j,\kappa}\, \xi_j$$

The coefficients here are to within the factor $\frac{1}{\Delta^2}$ the elements of the determinant obtained by multiplying the former

$$\left|A^{(\lambda)}_{\kappa,\nu}\right| \cdot \left|B^{(\mu)}_{j,\kappa}\right| = \Delta^{2p}\, \gamma_\lambda^{\left[\frac{p(p+1)}{2}\right]^2}\, \gamma_\mu^{\frac{p(p-1)(2p-1)}{6}}$$

We have thus expressed all of the p^3 quantities

$$\left[\lambda,\mu,\nu\right] \qquad \lambda,\mu,\nu = 0,1,2,\cdots\cdots p-1$$

as linear homogeneous functions of the p linearly independent ones

$$\left[0,0,\nu\right] \equiv \xi_\nu \qquad \nu = 0,1,\cdots\cdots p-1$$

Hence we see that every holomorphic function of x, y, z satisfying conditions (1) to (3) can be expressed as a linear homogeneous function of these p quantities. From which follows that there can be only p such functions which shall be linearly independent.

$\zeta_0, \zeta_1, \zeta_2, \ldots \zeta_i \ldots \zeta_{p-1}$

obviously satisfy the conditions

$$(20)\quad \begin{cases} \zeta_i(x, y, z) = [illegible](x, y, z + \frac{\omega_3}{p}) \\ \zeta_i(x, y, z+\omega_3) = \zeta_i(x, y, z) \\ \zeta_i(x, y, z + \frac{\omega_3}{p}) = \zeta_{i+1}(x, y, z) \end{cases}$$

Further if for brevity we write

$$a p^2 + 2\pi i\left(\frac{x}{\omega_1}p^3 + \frac{y}{\omega_2}p^2 + \frac{z}{\omega_3}p\right) = E(p)$$

and if we denote the substitution

$$\left(x, y, z;\ x + \frac{2\omega_1 a}{\pi i}p,\ y + \frac{[illegible]\omega_2 a}{[illegible]}p + \frac{3\omega_2 a}{\pi i}p^2,\ z + \frac{2\omega_3 y}{\omega_2}p + \frac{3\omega_3 x}{\omega_1}p^2 + \frac{2\omega_3 a}{\pi i}[illegible]\right)$$

by S_p we have also

$$(21)\quad \begin{cases} S_p\, \zeta_i(x, y, z) = e^{-E(p)}\, \zeta_i(x, y, z) \\ S_p\, \zeta_i(x, y, z) = e^{-E(p) - [illegible]\frac{\pi i}{p}}\, \zeta_i(x, y, z) \end{cases}$$

In general we have,

$$\lambda_{\lambda\mu\nu}(x,y,z) = \lambda\left(x+\frac{\lambda\omega_1}{p},\ y+\frac{\mu\omega_2}{p},\ z+\frac{\nu\omega_3}{p}\right)$$

$$\lambda_{\lambda\mu\nu}(x+\omega_1,y,z) = \lambda_{\lambda\mu\nu}(x,y+\omega_2,z) = \lambda_{\lambda\mu\nu}(x,y,z+\omega_3)$$

$$= \lambda_{\lambda\mu\nu}(x,y,z)$$

$$\lambda_{\lambda\mu\nu}\left(x+\frac{l\omega_1}{p},\ y+\frac{m\omega_2}{p},\ z+\frac{n\omega_3}{p}\right) = \lambda_{\lambda+l,\mu+m,\nu+n}(x,y,z)$$

$$S_p\,\lambda_{\lambda,\mu,\nu}(x,y,z) = \epsilon^{-E(\cdots)}\,\lambda_{\lambda\mu\nu}(x,y,z)$$

while the effect of the substitution S_ρ where $\rho < p$, is to change $\lambda_{\lambda,\mu,\nu}(x,y,z)$ into some other function altogether, in general.

Let us consider now, in connection with the function

$$(1)\quad \xi_0 = \varphi(x, y, z) = \sum_{m=-\infty}^{m=\infty} e^{am^4 + 2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)}$$

The functions obtained by increasing z by $\frac{\omega_3}{4}$, $\frac{\omega_3}{2}$, and by $\frac{3\omega_3}{4}$ respectively. One may write these briefly

$$(2)\quad \begin{cases} \varphi(x, y, z) = \varphi_0(x, y, z) = \sum_m e^{E(m)} \\ \varphi\left(x, y, z+\frac{\omega_3}{4}\right) = \varphi_1(x, y, z) = \sum_m i^m e^{E(m)} \\ \varphi\left(x, y, z+\frac{\omega_3}{2}\right) = \varphi_2(x, y, z) = \sum_m (-1)^m e^{E(m)} \\ \varphi\left(x, y, z+\frac{3\omega_3}{4}\right) = \varphi_3(x, y, z) = \sum (-i)^m e^{E(m)} \end{cases}$$

(From what has preceded, it is plain that the functions obtained by adding to x and to y respectively in (1) any multiples of the quarter periods corresponding to them, viz $\frac{\omega_1}{4}$ and $\frac{\omega_2}{4}$, will be linear homogeneous functions of the four functions (2). In particular

[illegible]

$$
(3)\begin{cases}
\varphi\left(x+\frac{\omega_1}{2}, y, z\right) = \varphi\left(x, y+\frac{\omega_2}{2}, z\right) = \varphi\left(x, y, z+\frac{\omega_3}{2}\right) \\
\text{and} \qquad\qquad\qquad\qquad\qquad = \varphi_2(x, y, z) \\
\varphi\left(x, y+\frac{\omega_2}{2}, z+\frac{\omega_3}{2}\right) = \varphi\left(x+\frac{\omega_1}{2}, y, z+\frac{\omega_3}{2}\right) = \varphi\left(x+\frac{\omega_1}{2}, y+\frac{\omega_2}{2}, z\right) \\
\qquad\qquad\qquad\qquad\qquad\qquad = \varphi_0(x, y, z)
\end{cases}
$$

Further we have manifestly

$$
(4)\begin{cases}
\varphi_j(x+\omega_1, y, z) = \varphi_j(x, y+\omega_2, z) = \varphi_j(x, y, z+\omega_3) \\
\qquad\qquad\qquad\qquad\qquad = \varphi_j(x, y, z) \\
S_1\, \varphi_j(x, y, z) = (-i)^j e^{-E(1)}\, \varphi_j(x, y, z) \\
S_0\, \varphi_j(x, y, z) = (-i)^{j} e^{-E(0)}\, \varphi_j(x, y, z)
\end{cases}
$$

$$j = 0, 1, 2, 3$$

If we apply the substitution

$$S_{\frac{1}{2}} = \left(x, y, z;\ x+\frac{2\omega_1 a}{2\pi i},\ y+\frac{3\omega_2 x}{2\omega_1}+\frac{3\omega_2 a}{4\pi i},\ z+\frac{[\text{illegible}]}{2\omega_2}+\frac{[\text{illegible}]}{4\omega_1}+\frac{[\text{illegible}]\,\omega_3}{8\pi i}\right)$$

we shall [illegible] the relations [illegible]

$$
(5)\begin{cases}
S_{\frac{1}{2}}\varphi_0(x, y, z) = e^{-E(\frac{1}{2})} \sum_m e^{a(m+\frac{1}{2})^4 + 2\pi i\left[\frac{x}{\omega_1}(m+\frac{1}{2})^3 + \frac{y}{\omega_2}(m+\frac{1}{2})^2 + \frac{z}{\omega_3}(m+\frac{1}{2})\right]} \\
S_{\frac{1}{2}}\varphi_1(x, y, z) = e^{-E(\frac{1}{2})} \sum_m i^m e^{a(m+\frac{1}{2})^4 + 2\pi i\left[\frac{x}{\omega_1}(m+\frac{1}{2})^3 + \frac{y}{\omega_2}(m+\frac{1}{2})^2 + \frac{z}{\omega_3}(m+\frac{1}{2})\right]} \\
S_{\frac{1}{2}}\varphi_2(x, y, z) = e^{-E(\frac{1}{2})} \sum_m (-1)^m e^{a(m+\frac{1}{2})^4 + 2\pi i\left[\frac{x}{\omega_1}(m+\frac{1}{2})^3 + \frac{y}{\omega_2}(m+\frac{1}{2})^2 + \frac{z}{\omega_3}(m+\frac{1}{2})\right]} \\
S_{\frac{1}{2}}\varphi_3(x, y, z) = e^{-E(\frac{1}{2})} \sum_m (-i)^m e^{a(m+\frac{1}{2})^4 + 2\pi i\left[\frac{x}{\omega_1}(m+\frac{1}{2})^3 + \frac{y}{\omega_2}(m+\frac{1}{2})^2 + \frac{z}{\omega_3}(m+\frac{1}{2})\right]}
\end{cases}
$$

(This suggests the following functions

which may be written [illegible]

$$(6)\quad \begin{cases} \psi_0(x, y, z) = \sum e^{E(u+z)} \\ \psi_1(x, y, z) = \sum i^{m} e^{E(u+z)} \\ \psi_2(x, y, z) = \sum (-1)^{m} e^{E(u+z)} \\ \psi_3(x, y, z) = \sum (-i)^{m} e^{E(u+z)} \end{cases}$$

Writing

$$e^{\frac{2}{7}\pi i} = \lambda \qquad e^{\frac{?}{7}\pi i} = \mu$$

we have at once

$$(7)\quad \begin{cases} \psi_j(x+\omega_1, y+\omega_2, z+\omega_3) = \lambda\,\psi_{j+1}(x, y, z) \\ \psi_j(x+\omega_1, \; y, \; z) = \mu\,\psi_{j+1}(x, y, z) \\ \psi_j(x, \; y+\omega_2, \; z) = i\,\psi_j(x, y, z) \\ \psi_j(x, \; y, \; z+\omega_3) = -\,\psi_j(x, y, z) \\ \psi_j(x, \; y, \; z+\frac{\omega_3}{7}) = \mu\,\psi_{j+1}(x, y, z) \end{cases}$$

$$j = 0, 1, 2, 3 \quad \text{and} \quad \psi_4 \equiv \psi_0$$

while $\psi_j(x+\frac{\omega_1}{2}, y, z)$ and $\psi_j(x+\frac{\omega_1}{7}, y, z)$ are entirely new functions not expressible as linear combinations of any of the functions ϕ_i or ψ_i.

It will be seen from these equations that the [illegible] of ψ_i [illegible]

of φ_3, for in the case of the [illegible]

$$(5)\quad \begin{cases} 8\,\omega_1 \text{ is the period corresponding to } x \\ [illegible]\,\omega_2 \quad " \quad " \quad " \quad y \\ [illegible]\,\omega_3 \quad " \quad " \quad " \quad z \end{cases}$$

although each of the substitutions

$$(8)\quad (x,y,z;\, x+4\omega_1, y, z),\ (x,y,z;\, x, y+2\omega_2, z),\ (x,y,z;\, x, y, z+\omega_3)$$

operating on φ_3 has only the effect of changing its sign.

The effect of the substitution S_1 on $\varphi_3(x,y,z)$ is the same as on $\varphi_1(x,y,z)$ [illegible]

$$(7)\quad \begin{cases} S_1\,\varphi_3(x,y,z) = [illegible]\, e^{-E(1)}\,\varphi_3(x,y,z) \\ S_1\,\varphi_3(x,y,z) = [illegible]\, e^{-E(\,)}\,\varphi_3(x,y,z) \end{cases}$$

[illegible]

$$S_{\frac{\tau}{2}}\,\varphi_3(x,y,z) = e^{-E(\tau)}\,\varphi_3(x,y,z)$$

[illegible]

$$(9)\quad \begin{cases} S_{\frac{\tau}{2}}\,\varphi_3(x,y,z) = [illegible]\, e^{-E(\frac{\tau}{2})}\,\varphi_3(x,y,z) \\ S_{\frac{1+\tau}{2}}\,\varphi_3(x,y,z) = (-i)^{(\,)}\, e^{-E(\frac{1+\tau}{2})}\,\varphi_3(x,y,z) \end{cases}$$

In general the effect of S_1 is to change $e^{E(m)}$ into $e^{E(m+1)}$, while S_τ [illegible] $e^{E(\,)}$

into $e^{E(m+1)}$. (Hence the effect of S_p on our functions is to within an exponential factor which may be taken out from under the sign of summation, the same as that of S_1^p.

Similarly $S_{\frac{1}{2}}$ changes $e^{E(m)}$ into $e^{E(m+\frac{1}{2})}$ and $S_{\frac{2r+1}{2}}$ changes $e^{E(m)}$ into $e^{E(m+r+\frac{1}{2})}$. Hence we see here also that, to within an exponential factor as above, the effect of S_1 is the same as that of $S_{\frac{1}{2}}^2$, and finally that of $S_{\frac{2r+1}{2}}$ the same as that of $S_{\frac{1}{2}}^{2r+1}$, or of $S_1^r S_{\frac{1}{2}}$, or of $S_r S_{\frac{1}{2}}$.

Changing x into $-x$ and z into $-z$ simultaneously has the effect of changing m into $-m$ in φ, and m into $-m-1$ in ψ; hence it leaves, to within the exponential

interchanges φ_1 and φ_3

changes ψ_1 into $-i\,\psi_3$

ψ_3 into $i\,\psi_1$

and ψ_2 into $-\psi_2$

Thus we see that, as regards x and z simultaneously

φ_0, φ_2, ψ_0, $\varphi_1\varphi_3$ and $\psi_1\psi_3$ are even

ψ_2 is odd

Changing the sign of x or of z only, or of y, changes the values of all the functions in such a way that no conclusions as to parity can be drawn in these cases.

In the case of each of these functions we may find those zeros which, like the zeros of the Θ-Functions cause the vanishing of the function by the cancellation in pairs of the terms of the series defining it. This we can accomplish by the examination of the function $\psi_2(x, y, z)$. ψ_2 [illegible] find

$$\psi_2(0, y, 0) = \sum_m (-1)^m e^{a(m+\frac{1}{2})^4 + 2\pi i \frac{y}{\omega_2}(m+\frac{1}{2})^2}$$

Changing m into $-m-1$, we get

$$\psi_2(0, y, 0) = \sum_m (-1)^{-m-1} e^{a(m+\frac{1}{2})^4 + 2\pi i \frac{y}{\omega_2}(m+\frac{1}{2})^2}$$

$$= -\sum_m (-1)^m e^{a(m+\frac{1}{2})^4 + 2\pi i \frac{y}{\omega_2}(m+\frac{1}{2})^2}$$

From which we see at once that

$$\Psi_2(0, y, 0) = 0$$

Or, more generally, from (8′) we have

$$\Psi_2(+h\omega_1, y, l\omega_3) = 0$$

where h and l are any integers [illegible]

[illegible] y [illegible]

Finally, applying the substitution S_q, $\Psi_2(x, y, z)$ is reproduced multiplied by the finite factor $(-1)^{q} e^{-\Xi(q)}$ which is different from zero. This gives us, then

$$\Psi_2\left(+2h\omega_1 + \frac{\omega_1 a}{\pi i} q,\ y + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2,\ l\omega_3 + \frac{2\omega_3 \gamma}{\omega_2} q + \frac{2\omega_3 a}{\pi i} q^3\right) = 0$$

The most general set of zeros of $\Psi_2(x, y, z)$ is then, without loss of generality, from (8′)

$$(11)\qquad \begin{cases} x = +2h\omega_1 + \dfrac{\omega_1 a}{\pi i} q \\ y = y + 2k\omega_2 + \dfrac{3\omega_2 a}{\pi i} q^2 \\ z = l\omega_3 + 2\dfrac{\omega_3 \gamma}{\omega_2} q + \dfrac{2\omega_3 a}{\pi i} q^3 \end{cases}$$

From the second and fifth equations of (7)

and the second equation of (10) we can write the following table of zeros, where, as above h, k, l, q, r are any integers

zero of	$x =$	$y =$	$z =$
φ_0	$(2h+1)\omega_1 + \frac{2\omega_1 a}{\pi i} q$ or $2h\omega_1 + \frac{2\omega_1 a}{\pi i} q$	$q + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2$ $q + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2$	$l\omega_3 + 2\frac{\omega_3 b}{\omega_2} q + \frac{2a\omega_3}{\pi i} q^3$ $(l+\frac{1}{2})\omega_3 + 2\frac{\omega_3 b}{\omega_2} q + \frac{2\omega_3 a}{\pi i} q^3$
ψ_1	$(2h+1)\omega_1 + \frac{2\omega_1 a}{\pi i} q$ or $2h\omega_1 + \frac{2\omega_1 a}{\pi i} q$	$q + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2$ $q + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2$	$l\omega_3 + 2\frac{\omega_3 b}{\omega_2} q + \frac{2\omega_3 a}{\pi i} q^3$ $(l+\frac{1}{2})\omega_3 + 2\frac{\omega_3 b}{\omega_2} q + \frac{2\omega_3 a}{\pi i} q^3$
ψ_2	$2h\omega_1 + \frac{2\omega_1 a}{\pi i} q$	$q + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2$	$l\omega_3 + 2q\frac{\omega_3 b}{\omega_2} q + \frac{2\omega_3 a}{\pi i} q^3$
ψ_3	$(2h+1)\omega_1 + \frac{2\omega_1 a}{\pi i} q$ or $2h\omega_1 + \frac{2\omega_1 a}{\pi i} q$	$q + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2$ $q + 2k\omega_2 + \frac{3\omega_2 a}{\pi i} q^2$	$l\omega_3 + 2q\frac{\omega_3 b}{\omega_2} q + \frac{2\omega_3 a}{\pi i} q^3$ $(l+\frac{1}{4})\omega_3 + 2q\frac{\omega_3 b}{\omega_2} + \frac{2\omega_3 a}{\pi i} q^3$
f_0	$h\omega_1 + \frac{2\omega_1 a}{\pi i}\left(\frac{2r+1}{2}\right)$	$q + k\omega_2 + \frac{3\omega_2 a}{\pi i}\left(\frac{2r+1}{2}\right)^2$	$(l+\frac{1}{2})\omega_3 + \frac{2\omega_3 b}{\omega_2}\left(\frac{2r+1}{2}\right) + \frac{2\omega_3 a}{\pi i}\left(\frac{2r+1}{2}\right)^3$
f_1	$h\omega_1 + \frac{2\omega_1 a}{\pi i}\left(\frac{2r+1}{2}\right)$	$q + k\omega_2 + \frac{3\omega_2 a}{\pi i}\left(\frac{2r+1}{2}\right)^2$	$(l+\frac{1}{4})\omega_3 + \frac{2\omega_3 b}{\omega_2}\left(\frac{2r+1}{2}\right) + \frac{2\omega_3 a}{\pi i}\left(\frac{2r+1}{2}\right)^3$
f_2	$h\omega_1 + \frac{2\omega_1 a}{\pi i}\left(\frac{2r+1}{2}\right)$	$q + k\omega_2 + \frac{3\omega_2 a}{\pi i}\left(\frac{2r+1}{2}\right)^2$	$l\omega_3 + \frac{2\omega_3 b}{\omega_2}\left(\frac{2r+1}{2}\right) + \frac{2\omega_3 a}{\pi i}\left(\frac{2r+1}{2}\right)^3$
f_3	$h\omega_1 + \frac{2\omega_1 a}{\pi i}\left(\frac{2r+1}{2}\right)$	$q + k\omega_2 + \frac{3\omega_2 a}{\pi i}\left(\frac{2r+1}{2}\right)^2$	$(l+\frac{3}{4})\omega_3 + \frac{2\omega_3 b}{\omega_2}\left(\frac{2r+1}{2}\right) + \frac{2\omega_3 a}{\pi i}\left(\frac{2r+1}{2}\right)^3$

Putting $u = v = w = s = r = 0$

we get the following simple zeros:-

Zeros of	$x =$	$y =$	$z =$
ψ_0	$2\omega_1$ or 0	y y	0 $\frac{\omega_3}{2}$
ψ_1	ω_1 or 0	y y	0 $\frac{\omega_3}{4}$
ψ_2	0	y	0
ψ_3	$2\omega_1$ or 0	y y	0 $\frac{3\omega_3}{4}$
φ_0	$\frac{\omega_1 u}{\pi i}$	$y - \frac{3}{4}\frac{\omega_2 u}{\pi i}$	$\frac{\omega_3}{2} + \frac{\omega_3 u}{4\pi i}$
φ_1	$\frac{\omega_1 u}{\pi i}$	$y + \frac{3}{4}\frac{\omega_2 u}{\pi i}$	$\frac{\omega_3}{4} + \frac{\omega_3 u}{4\pi i}$
φ_2	$\frac{\omega_1 u}{\pi i}$	$y + \frac{1}{4}\frac{\omega_2 u}{\pi i}$	$\frac{\omega_3 u}{4\pi i}$
φ_3	$\frac{\omega_1 u}{\pi i}$	$y + \frac{3}{4}\frac{\omega_2 u}{\pi i}$	$\frac{3\omega_3}{4} + \frac{\omega_3 u}{4\pi i}$

The zeros of $\varphi_0(x, y, z)$ might have been gotten directly as follows:-

$$\varphi_0(x, y, z) = \sum_{m=-\infty}^{m=\infty} e^{am^4 + 2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)}$$

$$= \sum_m e^{a(\mu-m)^4 + 2\pi i\left[\frac{x}{\omega_1}(\mu-m)^3 + \frac{y}{\omega_2}(\mu-m)^2 + \frac{z}{\omega_3}(\mu-m)\right]}$$

where μ is any integer.

The corresponding terms of these two series will be equal but of opposite sign for those values of x, y, z which make the exponents of e in the two cases differ by an odd multiple of πi for all values of m. Such values of x, y, z will evidently cause $\varphi_0(x,y,z)$ to vanish. We are to have then

$$a(\mu-m)^4 + 2\pi i\left[\frac{x}{\omega_1}(\mu-m)^3 + \frac{y}{\omega_2}(\mu-m)^2 + \frac{z}{\omega_3}(\mu-m)\right]$$
$$- \left[am^4 + 2\pi i\left(\frac{x}{\omega_1} m^3 + \frac{y}{\omega_2} m^2 + \frac{z}{\omega_3} m\right)\right] = (2n+1)\pi i$$

which on reduction becomes

$$(\mu-2m)\left\{\left(\frac{\pi i x}{\omega_1} + a\mu\right)(2m^2 - 2m\mu + \mu^2) + \left(\frac{\pi i x}{\omega_1}\mu^2 + \frac{2\pi i y}{\omega_2}\mu + \frac{2\pi i z}{\omega_3}\right)\right\} = (2n+1)\pi i$$

This condition is satisfied [illegible] for [illegible] many values, we have

$$(12)\quad \begin{cases} \mu \text{ any odd integer} \\ \dfrac{\pi i x}{\omega} + a\mu = \dfrac{\lambda\pi i}{2} \\ \dfrac{\pi i x}{\omega_1}\mu^2 + 2\dfrac{\pi i y}{\omega_2}\mu + \dfrac{2\pi i z}{\omega_3} = [\mu^2\lambda + 2(\lambda + 2\rho + 1)]\dfrac{\pi i}{2} \end{cases}$$

[illegible]

$$(13)\quad \begin{cases} x = \left(\dfrac{\lambda}{2} - \dfrac{a\mu}{\pi i}\right)\omega \\ y \quad \text{arbitrary} \\ z = \left(\dfrac{\lambda + 2\rho + 1}{2} - \dfrac{\mu y}{\omega_2} + \dfrac{a\mu^3}{2\pi i}\right)\omega_3 \end{cases}$$

We shall verify this result. We have.

$$\varphi_0\left[\left(\frac{\lambda}{2} - \frac{a\mu}{\pi i}\right)\omega_1, y, \left(\frac{\lambda + 2\rho + 1}{2} - \frac{\mu y}{\omega_2} + \frac{a\mu^3}{2\pi i}\right)\omega_3\right]$$

$$= \sum_m e^{am^4 + 2\pi i\left[\left(\frac{\lambda}{2} - \frac{a\mu}{\pi i}\right)m^3 + \frac{y}{\omega_2}m^2 + \left(\frac{\lambda + 2\rho + 1}{2} - \frac{\mu y}{\omega_2} + \frac{a\mu^3}{2\pi i}\right)m\right]}$$

$$= \sum_m e^{a(m^4 - 2m^3\mu + m\mu^3) + 2\pi i\frac{y}{\omega_2}m(m-\mu) + 2\pi i\left(\frac{\lambda}{2}m^3 + \frac{\lambda + 2\rho + 1}{2}m\right)}$$

$$= e^{-\frac{a\mu^4}{16}}\sum_m (-1)^m e^{a(m - \frac{\mu}{2})^4 - \frac{3}{2}a(m-\mu)\mu^2 m + 2\pi i\frac{y}{\omega_2}(m-\mu)m}$$

for $\qquad e^{\pi i(\lambda m^2 + \lambda + 2\rho + 1)} = (-1)^m$

Since for λ even $\lambda m^2 + \lambda + 2\rho + 1$ is odd

λ odd. $\lambda m^2 + \lambda + 2\rho + 1$ is even or odd

according as m is even or odd.

When m is replaced by $\mu - m$, the expression above is only altered by having $(-1)^m$ replaced by $(-1)^{\mu-m}$. Since μ is odd

$$(-1)^{\mu-m} = -(-1)^m$$

i.e. for the values (13) of the variables the function is equal to its negative, and must consequently be equal to zero.

In the above it was stated that y may be taken arbitrary. As a matter of fact either y or z may be so chosen, since these two variables are from (12) subjected only to the one condition

(4) [illegible]

or z by the one arbitrary, [illegible]

[illegible]

$$(5) \quad \begin{cases} x = \left(\frac{1}{2} - \text{[illegible]}\right) \omega \\ y = \left(\frac{1 + \text{[illegible]}}{\text{[illegible]}} - \text{[illegible]} - \text{[illegible]}\right) \text{[illegible]} \\ z \ \text{arbitrary} \end{cases}$$

For we have

$$\varphi_0\left[\left(\frac{\lambda}{2}-\frac{a\mu}{\pi i}\right)\omega_1,\ \left(\frac{\lambda+2\rho+1}{2\mu}+\frac{a\mu^2}{2\pi i}-\frac{z}{\mu\omega_3}\right)\omega_2,\ z\right]$$

$$=\sum_m e^{am^2+\pi i\left[\left(\frac{\lambda}{2}-\frac{a\mu}{\pi i}\right)m^3+\left(\frac{\lambda+2\rho+1}{2\mu}+\frac{a\mu^2}{2\pi i}-\frac{z}{\mu\omega_3}\right)m^2+\frac{z}{\omega_3}m\right]}$$

$$=e^{-\frac{a\mu^2}{4}}\sum_m e^{a\left(m-\frac{\mu}{2}\right)^2-\frac{a}{2}\mu^2 m(m-\mu)-\frac{2\pi i}{\mu\omega_3}z(m^2-\mu)+\pi i\lambda m^3+m^2\pi i\frac{\lambda+2\rho+1}{\mu}}$$

The effect of changing m into $\mu-m$ is to replace the factor $e^{\lambda m^3\pi i}$ in the above by

$$e^{[\lambda(\mu-m)^3+(\lambda+2\rho+1)\mu-2(\lambda+2\rho+1)m]\pi i}$$

which we may write for the sake of brevity

$$e^{I\pi i}$$

If λ is even, I is odd

λ is odd $e^{\lambda m^3\pi i}=(-1)^m$

and $e^{I\pi i}=(-1)^{\mu-m}=-(-1)^m$

so that for all values of λ

$$e^{I\pi i}=-e^{\lambda m^3\pi i}$$

from which we see at once as before that (15) is also a set of zeros.

The fact that z in (13) and in (11) contained the arbitrary quantity y might

have also assured us that we could so choose y as to give z any value we please and [illegible] [illegible] the [illegible], [illegible] of the function [illegible]

The zeros of $\varphi_1, \varphi_2, \varphi_3$ and those of ψ_1, ψ_2, ψ_3 may also be calculated directly in the same way we have just found those of φ_0, or they may be derived from those of φ_0 in a manner similar to that used in obtaining the zeros of all the rest from those of ψ_2.

A comparison of the sets of zeros obtained by the two methods, which in fact however are the same in principle, will manifestly show them to be identical if account is taken of (1).

By the first method the imaginary zeros were first obtained and from these we determined the most general zeros by

by observing what operations could be performed upon the function without altering its value except, perhaps, as to a finite factor other than zero.

By the method [illegible] the most general set of zeros at once. The simplest zeros are then gotten by putting

$$\lambda = \rho = 0 \qquad \mu = -1$$

I say that (13) for example is the most general set of zeros possible of the kind here considered. For, all the operations, as far as known, which leave the value of the function unaltered, or unaltered except as to a finite factor other than zero, are there provided for.

Thus λ so enters, that a change in it by an even integer amount corresponds to a change in x and z by some multiple of ω_1 and ω_3 respectively; while the change in λ will be an odd integer [illegible]

which cause the cancellation in pairs of the terms of the series, only y or z (but not both simultaneously) may be taken arbitrary, while x cannot be taken arbitrary at all since it enters alone in one of the equations of condition (12). No way of discovering other zeros has suggested itself, and it remains a question whether other zeros exist or all the zeros are confined within the above restrictions.

and z are increased or diminished by the same odd multiple of $\frac{\omega_1}{2}$ and $\frac{\omega_3}{2}$ respectively, which by (3), does not alter the value of the function.

The presence of ρ permits a change in z alone by any multiple of ω_3.

If μ is altered and the value of z so, that it be changed by an integer multiple of ω_2, z will be changed by an integer multiple of ω_3, and if altered by an odd multiple of $\frac{\omega_2}{2}$, the effect on z will be to change it by an odd multiple of $\frac{\omega_3}{2}$.

Finally, μ being an odd integer, it represents the result of the operation of S_b where b is any integer, the effect of which is, as we saw, to leave the value of the function unaltered, except as to a finite factor other than one.

In the [illegible] we have found [illegible]

The quotients

$$\frac{\varphi_0}{\varphi_2},\ \frac{\varphi_1}{\varphi_2},\ \frac{\varphi_3}{\varphi_2},\ \frac{\psi_0}{\varphi_2},\ \frac{\psi_1}{\varphi_2},\ \frac{\psi_2}{\varphi_2},\ \frac{\psi_3}{\varphi_2}$$

are doubly periodic, the substitutions

S_2 and $(x, y, z;\ x+\alpha\omega_1,\ y+\beta\omega_2,\ z+\gamma\omega_3)$ leaving $\frac{\varphi_0}{\varphi_2}$ unaltered

S_4 " $(x, y, z;\ x+\alpha\omega_1,\ y+\beta\omega_2,\ z+\gamma\omega_3)$ " $\frac{\varphi_1}{\varphi_2}$ and $\frac{\varphi_3}{\varphi_2}$ "

S_2 " $(x, y, z;\ x+8\alpha\omega_1,\ y+4\beta\omega_2,\ z+2\gamma\omega_3)$ " $\frac{\psi_0}{\varphi_2}$ "

S_4 " $(x, y, z;\ x+8\alpha\omega_1,\ y+4\beta\omega_2,\ z+2\gamma\omega_3)$ " $\frac{\psi_1}{\varphi_2}$ and $\frac{\psi_3}{\varphi_2}$ "

S_1 " $(x, y, z;\ x+8\alpha\omega_1,\ y+4\beta\omega_2,\ z+2\gamma\omega_3)$ " $\frac{\psi_2}{\varphi_2}$ "

α, β, γ being any integers.

If now we turn our attention to the derivatives of these quotients with respect to x, y or z and inquire whether analogously to the elliptic functions these derivatives are expressible in terms of any combination of the quotients themselves it would seem that such is not the case.

We shall first consider the derivative with respect to z. For convenience of reference the following table may be [illegible]

[illegible] That [illegible]

[illegible]

[illegible] is odd.

Also

[illegible]

$$\frac{\partial}{\partial z}\varphi_j(x+\omega_1, y, z) = \frac{\partial}{\partial z}\varphi_j(x, y+\omega_2, z) = \frac{\partial}{\partial z}\varphi_j(x, y, z+\omega_3)$$
$$= \frac{\partial}{\partial z}\varphi_j(x, y, z)$$

[illegible] the effect of adding multiples of $\omega_1, \omega_2, \omega_3$ to x, y, z respectively in $\frac{\partial}{\partial z}\psi_j(x, y, z)$ is the same as [illegible] the case of $\psi_j(x, y, z)$ [illegible]

[illegible]

$$S_1 \frac{\partial \psi_j}{\partial z} = [\text{illegible}]\, e^{-E_1} \left[\frac{\partial \psi_j}{\partial z} - [\text{illegible}]\right]$$
$$S_2 \frac{\partial \psi_j}{\partial z} = [\text{illegible}]\, e^{-E_2} \left[\frac{\partial \psi_j}{\partial z} - [\text{illegible}]\, \psi_j\right]$$
$$S_1' \frac{\partial \psi_j}{\partial z} = [\text{illegible}]\, e^{-E} \left[\frac{\partial \psi_j}{\partial z} - [\text{illegible}]\, \psi_j\right]$$
$$S_2' \frac{\partial \psi_j}{\partial z} = [\text{illegible}]\, e^{-E(z)} \left[\frac{\partial \psi_j}{\partial z} - [\text{illegible}]\, \psi_j\right]$$

$$j = [\text{illegible}]$$

[illegible]

$$\frac{\partial}{\partial z}\left[\frac{\psi_2}{\varphi_2}\right] = \frac{\varphi_2\frac{\partial}{\partial z}\psi_2 - \psi_2\frac{\partial}{\partial z}\varphi_2}{\varphi_2^2}$$

The numerator of this expression is holomorphic and, when operated on by S_1 is reproduced multiplied by $e^{-2E(1)}$. Hence if [illegible] rationally in terms of the φ_i and ψ_k $(i, k = 0, 1, 2, 3)$ it must be a homogeneous quadratic function of these.

Besides we [illegible] that

it is zero at x and z simultaneously

it is [illegible] when z is [illegible] by φ_3

it is reproduced multiplied by $e^{-2E(}$ when operated on by S_2.

Of all the 36 combinations of φ_i and ψ_k taken two at a time only ~~[illegible]~~ one $\varphi_0\psi_0$ satisfies all these requirements.

Moreover

$$\varphi_2\frac{\partial\psi_2}{\partial z} - \psi_2\frac{\partial\varphi_2}{\partial z}$$

is zero whenever φ_0 and whenever ψ_0 is.

[illegible] it would seem that we could write

$$[illegible]\frac{\partial [illegible]}{\partial z} - [illegible]\frac{\partial [illegible]}{\partial z} = A' [illegible]$$

where since this relation must hold when [illegible] on by S_1, and when x, y, z are [illegible]ered by multiples of $\omega_1, \omega_2, \omega_3$ respectively, A' is a constant, equal to

$$\frac{\varphi_2(0,0,0)\frac{\partial}{\partial z}\psi_2(0,0,0)}{\varphi_0(0,0,0)\ \psi_0(0,0,0)}$$

[illegible]

$$(1)\quad \frac{\partial}{\partial z}\left[\frac{\psi_2(x,y,z)}{\varphi_2(x,y,z)}\right] = \frac{\varphi_2(0,0,0)\frac{\partial}{\partial z}\psi_2(0,0,0)}{\varphi_0(0,0,0)\ \psi_0(0,0,0)}\ \frac{\varphi_0(x,y,z)\ [illegible]\ \psi_0(x,y,z)}{\varphi_2^2(x,y,z)}$$

[illegible] the substitution

$$(x, y, z;\ x+\omega_1, y, z)$$

we shall get

$$(2)\quad \frac{\partial}{\partial z}\left[\frac{\psi_1(x,y,z)}{\varphi_2(x,y,z)}\right] = \frac{\varphi_2(0,0,0)\frac{\partial}{\partial z}\psi_1(0,0,0)}{\varphi_0(0,0,0)\ \psi_0(0,0,0)}.\ \frac{\varphi_0(x,y,z)\ \psi_1(x,y,z)}{\varphi_2^2(x,y,z)}$$

[illegible]

Substituting the first set of zeros for ψ_3 given on page 23 this equation is satisfied. But

using the second set we get

$$\frac{\varphi_2(0,0,0)\frac{\partial}{\partial z}\psi_2(0,0,0)}{\varphi_3(0,0,0)\,\psi_1(0,0,0)} = \frac{\varphi_1(0,0,0)\frac{\partial}{\partial z}\psi_2(0,0,0)}{\varphi_2(0,0,0)\,\psi_1(0,0,0)}$$

which is manifestly not true for

$$\varphi_1(0,0,0) = \varphi_3(0,0,0)$$

while $\varphi_0(0,0,0) \neq \varphi_2(0,0,0)$

as may be seen from the definition of the functions.

Hence we must conclude that a solution of the form (1) does not exist.

Again, if we consider the numerator of

$$\frac{\partial}{\partial z}\left(\frac{\psi_0}{\varphi_2}\right) = \frac{\varphi_2\frac{\partial\psi_0}{\partial z} - \psi_0\frac{\partial\varphi_2}{\partial z}}{\varphi_2^2}$$

it will be found, that of all the combinations [illegible] $\varphi_0\psi_2$ [illegible] the conditions that it does. [illegible] whenever either of φ_0 or ψ_2 does.

But on writing

(3) $$\frac{\partial}{\partial z}\left[\frac{\varphi_0(x,y,z)}{\psi_2(x,y,z)}\right] = F\,\frac{\varphi_0(x,y,z)\,\psi_2(x,y,z)}{\psi_2^2(x,y,z)}$$

where as before B can only be a constant, we shall find, on substituting the first set of zeros for $\varphi_2(x, y, z)$ given on page 20

$$B = \frac{\varphi_2(0,0,0)\frac{\partial}{\partial z}\psi_2(0,0,0)}{\varphi_0(0,0,0)\,\psi_0(0,0,0)}$$

while substituting the second set of zeros for $\varphi_0(x, y, z)$ we find

$$B = \frac{\varphi_0(0,0,0)\frac{\partial}{\partial z}\psi_2(0,0,0)}{\varphi_2(0,0,0)\,\psi_0(0,0,0)}$$

But these two values are not the same since

$$\varphi_2^2(0,0,0) \neq \varphi_0^2(0,0,0)$$

So that we conclude that no relation of the form (3) exists.

In the same way, the expressions for the derivatives of all the various quotients of a ψ by a φ in terms of the quotients fail to [illegible] and the tests at our disposal.

The difficulty seems to lie in the fact that the periods of ψ are greater, exactly [illegible]

[illegible] of Q. The
we can pass from γ_i to γ_{i+1} by either [illegible] the substitutions

[illegible]

[illegible] S_2 [illegible] γ_{i+1} [illegible] relations [illegible] (c) [illegible]

It should be noted that this test does not apply in the case of the Θ- and Ξ-functions.

As no general theorem analogous to that made use of in this connection in the case of the Θ-functions could be established for the functions here considered the one method [illegible] to [illegible]

such relations exist.

In a similar way, no quadratic relations between ~~any of~~ the φ_λ and ψ_λ satisfying all the tests at our command could be found.

The objection, above mentioned, as holding good against the relations between the derivatives of the quotients and the quotients themselves do not seem to hold in the case where only the φ_λ or the ψ_λ separately are involved.

Thus it was found that

$$(4)\quad \frac{\partial}{\partial z}\left[\frac{\varphi_0(x,y,z)}{\varphi_2(x,y,z)}\right] = \frac{\varphi_2(x,y,z)\frac{\partial}{\partial z}\varphi_0(x,y,z) - \varphi_0(x,y,z)\frac{\partial}{\partial z}\varphi_2(x,y,z)}{\varphi_2^2(x,y,z)}$$

$$= \frac{\psi_0(0,0,0)\frac{\partial}{\partial z}\psi_2(0,0,0)}{\psi_3^2(0,0,0) - \psi_1^2(0,0,0)} \cdot \frac{\varphi_1^2(x,y,z) - \varphi_0^2(x,y,z)}{\varphi_2^2(x,y,z)}$$

and

$$(5)\quad \frac{\partial}{\partial z}\left[\frac{\psi_0(x,y,z)}{\psi_2(x,y,z)}\right] = \frac{\psi_2(x,y,z)\frac{\partial}{\partial z}\psi_0(x,y,z) - \psi_0(x,y,z)\frac{\partial}{\partial z}\psi_2(x,y,z)}{\psi_2^2(x,y,z)}$$

$$= \frac{\psi_0(0,0,0)\frac{\partial}{\partial z}\psi_2(0,0,0)}{\psi_3^2(0,0,0) - \psi_1^2(0,0,0)} \cdot \frac{\psi_1^2(x,y,z) - \psi_0^2(x,y,z)}{\psi_2^2(x,y,z)}$$

[illegible] the remark [illegible]

the iteration $S_{\frac{1}{2}}$. Together with the other relations derived from them by all the operations at our command, satisfy all the tests that were applied to them.

These last results are given here not as proved relations but as such which have not been disproved.

The derivatives with respect to x and y are more complicated than those with respect to z. Hence it will be at least no easier to establish relations involving the former, than to establish any involving the latter. Thus

$$S_1 \frac{\partial \varphi_0}{\partial y} = \frac{2\pi i}{\omega_2} e^{-E(1)} \sum_m m^2 e^{E(m+1)}$$

$$= \frac{2\pi i}{\omega_2} e^{-E(1)} \sum_m (m-1)^2 e^{E(m)}$$

$$= e^{-E(1)} \left[\frac{\text{[illegible]}}{y} - \frac{\text{[illegible]}}{\omega_2} \text{[illegible]} + \frac{\text{[illegible]}}{\omega_2} \right]$$

Similarly

$$S_1 \frac{\partial \text{[illegible]}}{\partial x} = e^{-E(1)} \left[\text{[illegible]} \right]$$

[illegible]

$$\Phi_0 = [illegible], \quad \Phi_1 = [illegible], \quad \Psi_0 = [illegible], \quad \Psi_1 = [illegible]$$

[illegible]

[illegible]

$\Phi_0, \ \Phi_1, \ \Psi_1$ [illegible]

Ψ_0 [illegible]

$$\begin{cases} S_1 \Phi_0 = -e^{-2E\omega} \Phi_0 \\ S_1 \Phi_1 = e^{-2E\omega} \Phi_1 \\ S_1 \Psi_0 = -e^{-2E\omega} \Psi_0 \\ S_1 \Psi_1 = e^{-2E\omega} \Psi_1 \end{cases} \qquad \begin{cases} S_{\frac{1}{2}} \Phi_0 = e^{-E\omega} \Phi_0 \\ S_{\frac{1}{2}} \Phi_1 = e^{-E\omega} \Phi_1 \\ S_{\frac{1}{2}} \Psi_0 = -e^{-E\omega} \Psi_0 \\ S_{\frac{1}{2}} \Psi_1 = e^{-E\omega} \Psi_1 \end{cases}$$

$$\begin{cases} \Phi_j(x+\omega_1, y+\omega_2, z+\omega_3) = \Phi_j(x, y, z) \\ \Phi_j(x+\omega_1, y, z) = \Phi_j(x, y, z) & j = 0, 1 \\ \Phi_j(x, y+\omega_2, z) = \Phi_j(x, y, z) \\ \Phi_j(x, y, z+\omega_3) = \Phi_j(x, y, z) \end{cases}$$

$$\begin{cases} \Psi_j(x+\omega_1, y+\omega_2, z+\omega_3) = -i\,\Psi_{j+1}(x, y, z) \\ \Psi_j(x+\omega_1, y, z) = i\,\Psi_{j+1}(x, y, z) & j = 0, 1 \\ \Psi_j(x, y+\omega_2, z) = -\Psi_j(x, y, z) & \Psi_2 \equiv \Psi_0 \\ \Psi_j(x, y, z+\omega_3) = \Psi_j(x, y, z) \\ \Psi_j(x, y, z+\frac{\omega_3}{2}) = i\,\Psi_{j+1}(x, y, z) \end{cases}$$

[illegible] $\overline{\Phi}_1\left(u+\frac{\omega_3}{2}\right)$ [illegible]

$\overline{\Psi}_1\left(u+\frac{\omega_3}{2}\right)$ [illegible]

[illegible] $\overline{\Phi}_0, \overline{\Phi}_1, \overline{\Psi}_0, \overline{\Psi}_1$

[illegible]

$$\frac{\partial}{\partial z}\overline{\Phi}_0 = [illegible]$$

$$S_1 \frac{\partial}{\partial z}\overline{\Phi}_0 = -e^{-2E(t)}\left[[illegible] \right]$$

[illegible]

$$\begin{cases} S_1 \dfrac{\partial \overline{\Phi}_0}{\partial z} = -e^{-2E(t)}\left[\dfrac{\partial \overline{\Phi}_0}{\partial z} - [illegible]\,\overline{\Phi}_0\right] \\ S_1 \dfrac{\partial \overline{\Phi}_1}{\partial z} = e^{-2E(t)}\left[\dfrac{\partial \overline{\Phi}_1}{\partial z} - [illegible]\,\overline{\Phi}_1\right] \\ S_1 \dfrac{\partial \overline{\Psi}_0}{\partial z} = -e^{-2E(t)}\left[\dfrac{\partial \overline{\Psi}_0}{\partial z} - [illegible]\,\overline{\Psi}_0\right] \\ S_1 \dfrac{\partial \overline{\Psi}_1}{\partial z} = e^{-2E(t)}\left[\dfrac{\partial \overline{\Psi}_1}{\partial z} - [illegible]\,\overline{\Psi}_1\right] \end{cases}$$

[illegible]

and

$$\begin{cases} S_2 \dfrac{\partial \overline{\Phi}_0}{\partial z} = e^{-2E(t)}\left[\dfrac{\partial \overline{\Psi}_0}{\partial z} - \dfrac{2\pi i}{\omega_3}\,\overline{\Psi}_0\right] \\ S_2 \dfrac{\partial \overline{\Phi}_1}{\partial z} = e^{-2E(t)}\left[\dfrac{\partial \overline{\Psi}_1}{\partial z} - \dfrac{2\pi i}{\omega_3}\,\overline{\Psi}_1\right] \\ S_2 \dfrac{\partial \overline{\Psi}_0}{\partial z} = -e^{-2E(t)}\left[\dfrac{\partial \overline{\Phi}_0}{\partial z} - [illegible]\,\overline{\Phi}_0\right] \\ S_2 \dfrac{\partial \overline{\Psi}_1}{\partial z} = e^{-2E(t)}\left[\dfrac{\partial \overline{\Phi}_1}{\partial z} - [illegible]\,\overline{\Phi}_1\right] \end{cases}$$

[illegible]

[illegible] the [illegible] [illegible] [illegible] [illegible] of the quotient of any two of our functions $\Phi_0, \Phi_1, \Psi_0, \Psi_1$ in terms of all the quotients we shall meet with the same [illegible] [illegible]. Thus if we form the derivative

$$\frac{\partial}{\partial z}\left(\frac{\Psi_0}{\Phi_0}\right) = \frac{\Phi_0 \frac{\partial \Psi_0}{\partial z} - \Psi_0 \frac{\partial \Phi_0}{\partial z}}{\Phi_0^2}$$

we shall find that $\Phi_1 \Psi_1$ only, of all the combinations of our functions, behaves exactly like $\Phi_0 \frac{\partial \Psi_0}{\partial z} - \Psi_0 \frac{\partial \Phi_0}{\partial z}$ when put to the test of all the above operations, and besides, this numerator vanishes for all the known zeros of Φ_1 and of Ψ_1. Thus we write,

$$(6) \qquad \frac{\partial}{\partial z}\left[\frac{\Psi_0(x, y, z)}{\Phi_0(x, y, z)}\right] = C \, \frac{\Phi_1(x, y, z)\, \Psi_1(x, y, z)}{\Phi_0^2(x, y, z)}$$

where, as before C must be a constant, we shall find that, according as we use the first set of zeros or the second set of zeros of $\Psi_0(x, y, z)$, which are those

[illegible]

$$C = \frac{\Phi_0(0,0,0)\,\frac{\partial}{\partial z}\,\Phi_1(0,0,0)}{\Phi_1(0,0,0)\,\Psi_1(0,0,0)}$$

or

$$C = \frac{\Phi_1(0,0,0)\,\frac{\partial}{\partial z}\,\Phi_0(0,0,0)}{\Phi_0(0,0,0)\,\Psi_0(0,0,0)}$$

But these two are not the same since

$$\Phi_0^2(0,0,0) \neq \Phi_1^2(0,0,0)$$

Hence we conclude that no relations of the type (6) exist.

(7) Quadratic relations between $\Phi_0, \Phi_1, \Psi_0, \Psi_1$ also seem not to exist for the same reason viz. because the periods of Φ_0 and Φ_1 are smaller than those of Ψ_0 and Ψ_1.

It may be mentioned in this connection that the following symmetrical quartic relation between the functions $\Phi_0, \Phi_1, \Psi_0, \Psi_1$ [illegible] [illegible] in the course of the work [illegible] [illegible] [illegible]

The conditions involved; viz

$$I_0^2 \Phi^2 - I_1^2 \Phi^2 = A\left[I_1^{-1} + \Phi_1^{-1} - \Psi_1^{-1} - \Psi_1^{-1} \right]$$

where A is a constant whose value [illegible]

[illegible]

Consider the holomorphic function

$$f_1(x,y,z) = \prod_{k=0}^{k=\infty}\left(1+e^{\text{[illegible]}}\right)$$

where the real part of a is negative.

If, as before, we denote the substitution

$$\left(x, y, z;\ x+\frac{2a\omega_1}{\pi i},\ y+\frac{3\omega_2 x}{\omega_1}+\frac{3a\omega_2}{\pi i},\ z+\frac{2\omega_3 y}{\omega_2}+\frac{3\omega_3 x}{\omega_1}+\frac{2a\omega_3}{\pi i}\right)$$

by S, it is obvious that

$$S f_1(x,y,z) = \frac{1}{1+e^{2a+2\pi i\left(\frac{3x}{\omega_1}+\frac{4y}{\omega_2}+\frac{4z}{\omega_3}\right)}}\, f_1(x,y,z)$$

Again writing

$$f_2(x,y,z) = \prod_{k=0}^{k=\infty}\left(1+e^{2a(2k+1)^3-2\pi i\left[\frac{3x}{\omega_1}(2k+1)^2-\frac{4y}{\omega_2}(2k+1)+\frac{4z}{\omega_3}\right]}\right)$$

we see at once that

$$S f_2(x,y,z) = \left[1+e^{-2a-2\pi i\left(\frac{3x}{\omega_1}+\frac{4y}{\omega_2}+\frac{4z}{\omega_3}\right)}\right] f_2(x,y,z)$$

Finally, writing

$$F(x,y,z) = f_1(x,y,z)\cdot f_2(x,y,z)$$

we have

$$S F(x,y,z) = \frac{1-e^{-2a-2\pi i\left(\frac{3x}{\omega_1}+\frac{4y}{\omega_2}+\frac{4z}{\omega_3}\right)}}{1+e^{2a+2\pi i\left(\frac{3x}{\omega_1}+\frac{4y}{\omega_2}+\frac{4z}{\omega_3}\right)}}\cdot F(x,y,z)$$

$$S\,\Xi(x,y,z) = e^{-2xz-2\pi i\left(\frac{x}{\omega_1}+\frac{y}{\omega_2}+\frac{z}{\omega_3}\right)}\,\Xi(x,y,z)$$

We also have

$$\Xi\left(x+\frac{\omega_1}{3},y,z\right) = \Xi\left(x,y+\frac{\omega_2}{4},z\right) = \Xi\left(x,y,z+\frac{\omega_3}{[illegible]}\right)$$
$$= \Xi(x,y,z)$$

If we put

$$[illegible] = A,\quad \frac{\omega_1}{3} = \Omega_1,\quad \frac{\omega_2}{4} = \Omega_2,\quad \frac{\omega_3}{4} = \Omega_3$$

our function becomes

$$\Sigma(x,y,z) = \prod_k\left[\left(1+e^{A(2k+1)^3+2\pi i\left[\frac{x}{\Omega_1}(2k+1)^2+\frac{y}{\Omega_2}(2k+1)+\frac{z}{\Omega_3}\right]}\right)\times\left(1+e^{A(2k+1)^3-2\pi i\left[\frac{x}{\Omega_1}(2k+1)^2-\frac{y}{\Omega_2}(2k+1)+\frac{z}{\Omega_3}\right]}\right)\right]$$

Then we have

$$\Sigma(x+\Omega_1,y,z) = \Sigma(x,y+\Omega_2,z) = \Sigma(x,y,z+\Omega_3)$$
$$= \Sigma(x,y,z)$$

and denoting by T the transformation [illegible] of [illegible]

$$\left(x,y,z;\ x+\frac{3\Omega_1 A}{\pi i},\ y+\frac{4\Omega_2 x}{\Omega_1}+\frac{[illegible]\Omega_2 A}{\pi i},\ z+\frac{[illegible]\Omega_3 y}{\Omega_2}+\frac{[illegible]\Omega_3 x}{\Omega_1}+\frac{[illegible]\Omega_3 A}{\pi i}\right)$$

we have

$$T\Sigma(x,y,z) = e^{-A-2\pi i\left(\frac{x}{\Omega_1}+\frac{y}{\Omega_2}+\frac{z}{\Omega_3}\right)}\,\Sigma(x,y,z)$$

The function $\Sigma(x,y,z)$ which resembles the functions already considered in being [illegible]

and in being reproduced to within a factor on being subjected to a linear substitution, seem to differ from them in not satisfying any simple differential equation or equations.

There seems also to be more freedom in obtaining the zeros of this function. Thus while in the case of the functions already considered, only y or z separately could be taken arbitrary, then the remaining [illegible] fixed by one condition, and x had to be chosen subject to two such [illegible] conditions, [illegible] function $\Sigma(x, y, z)$ vanishes whenever either of the following conditions is satisfied:-

$$A(2k+1)^3 + 2\pi i\left[\frac{x}{\Omega_1}(2k+1)^2 + \frac{y}{\Omega_2}(2k+1) + \frac{z}{\Omega_3}\right] = (2l+1)\pi i$$

or

$$A(2k+1)^3 - 2\pi i\left[\frac{x}{\Omega_1}(2k+1)^2 - \frac{y}{\Omega_2}(2k+1) + \frac{z}{\Omega_3}\right] = (2l+1)\pi i$$

where l is any integer, positive, zero or negative and k is any positive integer, including zero.

The [illegible]

is made the most striking by noticing that the zeros we found for $\phi_1(x, y, z)$ are also such for $\Sigma(x, y, z)$. But all the zeros of the latter are not included in the former.

It may be interesting to note still further the similarities and the dissimilarities existing between our two classes of functions.

For the sake of brevity write

$$E_1(\kappa) = A(2\kappa+1)^3 + 2\pi i\left[\frac{x}{\Omega_1}(2\kappa+1)^2 + \frac{y}{\Omega_2}(2\kappa+1) + \frac{z}{\Omega_3}\right]$$

$$E_2(\kappa) = A(2\kappa+1)^3 - 2\pi i\left[\frac{x}{\Omega_1}(2\kappa+1)^2 - \frac{y}{\Omega_2}(2\kappa+1) + \frac{z}{\Omega_3}\right]$$

Increasing z by $\frac{\Omega_3}{2}$ we get a new function

$$\Sigma\left(x, y, z+\frac{\Omega_3}{2}\right) \equiv \Sigma_1(x, y, z) = \prod_\kappa\left[\left(1-e^{E_1(\kappa)}\right)\left(1-e^{E_2(\kappa)}\right)\right]$$

Then we have

$$\Sigma\left(x+\frac{\Omega_1}{2}, y, z\right) = \Sigma\left(x, y+\frac{\Omega_2}{2}, z\right) = \Sigma\left(x, y, z+\frac{\Omega_3}{2}\right)$$

$$= \Sigma\left(x+\frac{\Omega_1}{2}, y+\frac{\Omega_2}{2}, z+\frac{\Omega_3}{2}\right) = \Sigma_1(x, y, z)$$

$$\Sigma\left(x, y+\frac{\Omega_2}{2}, z+\frac{\Omega_3}{2}\right) = \Sigma\left(x+\frac{\Omega_1}{2}, y, z+\frac{\Omega_3}{2}\right) = \Sigma\left(x+\frac{\Omega_1}{2}, y+\frac{\Omega_2}{2}, z\right)$$

$$= \Sigma(x, y, z) = \Sigma_0(x, y, z)$$

The effect of the substitution

$$T_p = ([illegible])$$

for p any quantity, is to change

$$\Xi_1(\lambda) \text{ into } \Xi_1(\lambda+p)$$

$$\Xi_2(\lambda) \text{ into } \Xi_2(\lambda-p)$$

$$\therefore T_p \Sigma_0(x, y, z) = \frac{[1+e^{-\Xi_1(0)}][1+e^{-\Xi_1(1)}] \cdots\cdots [1+e^{-\Xi_1(p-1)}]}{[1+e^{\Xi_1(0)}][1+e^{\Xi_1(1)}] \cdots\cdots [1+e^{\Xi_1(p-1)}]} \Sigma_0(x, y, z)$$

$$= e^{-[\Xi_1(0)+\Xi_1(1)+ \cdots\cdots + \Xi_1(p-1)]} \Sigma_0(x, y, z)$$

Similarly it is readily seen that

$$T_p \Sigma_1(x, y, z) = (-1)^p e^{-[\Xi_1(0)+\Xi_1(1)+ \cdots\cdots + \Xi_1(p-1)]} \Sigma_1(x, y, z)$$

The effect of the substitution

$$T_{\frac{1}{2}} = \left(x, y, z;\ x + \frac{[illegible]\,\Delta}{2\pi i},\ y + \frac{[illegible]}{\Omega_1} + \frac{[illegible]\,\Delta}{2\pi i},\ z + \frac{[illegible]}{\Omega_2} + \frac{[illegible]}{\Omega_1} + \frac{[illegible]\,\Delta}{2\pi i}\right)$$

is to change

$$\Xi_1(\lambda) \text{ into } \Xi_1(\lambda+\tfrac{1}{2})$$

$$\Xi_2(\lambda) \text{ into } \Xi_2(\lambda-\tfrac{1}{2})$$

thus giving rise [illegible] to entirely new functions

$$T_{\frac{1}{2}} \Sigma_0(x, y, z) \equiv \Xi_0(x, y, z) = \prod_\lambda \left[\left(1 + e^{\Xi_1(\lambda+\frac{1}{2})}\right)\left(1 + e^{\Xi_2(\lambda-\frac{1}{2})}\right)\right]$$

$$= \prod_\lambda \left[\left(1 + e^{\Delta(2\lambda+2)^3 + 2\pi i\left[\frac{x}{\Omega_1}(2\lambda+2)^2 + \frac{y}{\Omega_2}(2\lambda+2) + \frac{z}{\Omega_3}\right]}\right)\right.$$

$$\left.\times \left(1 + e^{[illegible]}\right)\right]$$

$$T_{\frac{1}{2}} \Sigma_1(x, y, z) \equiv \Xi_1(x, y, z) = \prod \left[\left(1 - e^{E_1(x+1)}\right)\left(1 - e^{E_1(x-1)}\right) \right]$$

It is noticeable that instead of the periods of Ξ being greater than those of Σ we have

$$\Xi\left(x + \frac{\Omega_1}{2}, y, z\right) = \Xi\left(x, y + \frac{\Omega_2}{2}, z\right) = \Xi(x, y, z + \Omega_3)$$
$$= \Xi(x, y, z)$$

Also

$$\Xi_0\left(x, y, z + \frac{\Omega_3}{2}\right) = \Xi_1(x, y, z)$$
$$\Xi_1\left(x, y, z + \frac{\Omega_3}{2}\right) = \Xi_0(x, y, z)$$

but there seems no way of passing from $\Xi_0(x, y, z)$ to $\Xi_1(x, y, z)$ or vice versa, by a change in x or y.

Again, as in the case of $\Sigma(x, y, z)$, we have

$$T_p \Xi_0(x, y, z) = e^{-\left[E_1(\frac{1}{2}) + E_1(\frac{3}{2}) + \cdots\cdots + E_1(p - \frac{1}{2})\right]} \Xi_0(x, y, z)$$
$$T_p \Xi_1(x, y, z) = (-1)^p e^{-\left[E_1(\frac{1}{2}) + E_1(\frac{3}{2}) + \cdots\cdots + E_1(p - \frac{1}{2})\right]} \Xi_1(x, y, z)$$

Also

$$T_{\frac{1}{2}} \Xi_0(x, y, z) = \prod_0^\infty \left[\left(1 + e^{E_1(x+1)}\right)\left(1 + e^{E_2(x-1)}\right)\right]$$
$$= e^{-E_1(0)} \Sigma_0(x, y, z) = T_1 \Sigma_0(x, y, z)$$

$$T_{\frac{1}{2}} \Xi_1(x, y, z) = -e^{-E_1(0)} \Sigma_1(x, y, z) = T_1 \Sigma_1(x, y, z)$$

But we had, by definition

$$T_{\frac{1}{2}}\Sigma_0(x,y,z) = \Xi_0(x,y,z) \qquad T_{\frac{1}{2}}\Sigma_1(x,y,z) = \Xi_1(x,y,z)$$

Hence we see that

$$T_{\frac{1}{2}}^{2}\Sigma(x,y,z) = T_1\Sigma(x,y,z)$$

and similarly

$$T_{\frac{1}{2}}^{2}\Xi(x,y,z) = T_1\Xi(x,y,z)$$

i.e. the effect of two successive operations of $T_{\frac{1}{2}}$ is identical to that of a single application of T_1.

Changing the sign of x and z simultaneously interchanges E_1 and E_2. From this follows that Σ_0 and Σ_1 are even as to x and z simultaneously.

But, for Ξ we have the values changed, thus

$$\Xi_0(-x,y,-z) = \frac{1+e^{E_1(-\frac{1}{2})}}{1+e^{E_2(-\frac{1}{2})}}\,\Xi_0(x,y,z)$$

$$\Xi_1(-x,y,-z) = \frac{1-e^{E_1(-\frac{1}{2})}}{1-e^{E_2(-\frac{1}{2})}}\,\Xi_1(x,y,z)$$

By considering, it may be [illegible] to [illegible] that by taking the logarithmic derivative of $\Sigma_0(x,y,z)$ with respect to z we shall obtain a new function analogous to the

Σ-function of one variable. Thus, writing

$$\Sigma_0(x,y,z) = \prod_{k} \Big(1 + e^{2A(2k+1)^3 + [illegible]} \ldots$$

$$+ 2e^{A(2k+1)^3 + 2\pi i \frac{y}{\Omega_2}(2k+1)} \cos\left[2\pi \frac{x}{\Omega_1}(2k+1)^2 + \frac{z}{\Omega_3}\right]\Big)$$

we have $$\frac{\frac{\partial}{\partial z}\Sigma_0(x,y,z)}{\Sigma_0(x,y,z)} = \Lambda(x,y,z) =$$

$$-\frac{2}{\Omega_3}\sum_{k=0}^{k=\infty} \frac{e^{A(2k+1)^3 + 2\pi i \frac{y}{\Omega_2}(2k+1)} \sin\left[2\pi \frac{x}{\Omega_1}(2k+1)^2 + \frac{z}{\Omega_3}\right]}{1 + e^{2A(2k+1)^3 + 4\pi i \frac{y}{\Omega_2}(2k+1)} + [illegible]\, e^{A(2k+1)^3 + 2\pi i \frac{y}{\Omega_2}(2k+1)} [illegible] \left[2\pi \frac{x}{\Omega_1}(2k+1)^2 + \frac{z}{\Omega_3}\right]}$$

(This series is uniformly convergent, since the real part of A is negative, and therefore represents a function.

$\Lambda(x,y,z)$ [illegible]

$$\Lambda(x+\Omega_1, y, z) = \Lambda(x, y+\Omega_2, z) = \Lambda(x, y, z+\Omega_3) = \Lambda(x,y,z)$$

[illegible]

$$\frac{\partial}{\partial x}\Lambda(x,y,z) = \frac{[illegible]\, e^{-A-2\pi i\left(\frac{x}{\Omega_1}+\frac{y}{\Omega_2}+\frac{z}{\Omega_3}\right)}}{e^{-A-2\pi i\left(\frac{x}{\Omega_1}+\frac{y}{\Omega_2}+\frac{z}{\Omega_3}\right)}} + \Lambda(x,y,z)$$

$$= -\frac{2\pi i}{\Omega_3} + \Lambda(x,y,z)$$

Another differentiation will give us a [illegible]

$$\frac{[illegible]}{[illegible]} = \frac{[illegible]}{\partial z} = \frac{[illegible]}{\partial z} = \frac{[illegible]}{\partial z}$$

[illegible]

$$T_1 \frac{\partial A(x, y, z)}{\partial z} = \frac{\partial A(x, y, z)}{\partial z}$$

(The successive derivatives also have the same property.)

[illegible] Sketch.

[illegible]
[illegible]
[illegible] School,
where he was enrolled from 1877 till 1883. He
then entered the Baltimore City College, and
upon graduation in 1885 was admitted as a
candidate for the degree of Bachelor of Arts,
in the Johns Hopkins University. This
degree was conferred upon him in June
[illegible]
[illegible]
the degree of Doctor of Philosophy, [illegible]
[illegible]
[illegible]
[illegible]
Upon receiving his Bachelor's degree he
[illegible]
[illegible]

Assistant in Mathematics. This position he held [illegible]

[illegible]

the Fellowship in Mathematics.